Hankel operators on Bergman spaces.

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Interuniversity Geometry Seminar

2022-04-08

Introduction

Let $\mathrm{Hol}(\mathbb{D})$ be the space of all holomorphic functions on the unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$ and let dA denote the normalized Lebesgue area measure on $\mathbb{D}$. The standard Bergman space A^2_α, $\alpha > -1$, is given by

$$\mathrm{A}^2_\alpha = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \ \|f\|_\alpha := \left(\int_{\mathbb{D}} |f(z)|^2 dA_\alpha(z) \right)^{1/2} < \infty \right\},$$

where $dA_\alpha(z) := (\alpha+1)(1-|z|^2)^\alpha dA(z)$. Recall that A^2_α is a reproducing Kernel Hilbert space with the kernel

$$K(z,w) = \frac{1}{(1-\overline{w}z)^{\alpha+2}}, \quad z,w \in \mathbb{D}.$$

The orthogonal projection from $L^2_\alpha := L^2(\mathbb{D}, dA_\alpha)$ onto A^2_α will be denoted by P_α.

$$P_\alpha f(z) = \int_{\mathbb{D}} f(w)K(z,w)dA_\alpha(w) = \int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^{\alpha+2}} dA_\alpha(w).$$

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Let $\phi \in \mathrm{A}^2_\alpha$. The linear transformation $H_{\overline{\phi}} = M_{\overline{\phi}} - P_\alpha M_{\overline{\phi}}$

$$H_{\overline{\phi}} f = \overline{\phi} f - P_\alpha(\overline{\phi} f),$$

is densely defined operator from A^2_α into $L^2_\alpha \ominus \mathrm{A}^2_\alpha$ which is called the (big) Hankel operator with symbol $\overline{\phi}$. An integral formula of $H_{\overline{\phi}}$ is

$$\begin{aligned} H_{\overline{\phi}} f(z) &= \overline{\phi}(z)\langle f, K_z\rangle - \langle \overline{\phi} f, K_z\rangle \\ &= \int_{\mathbb{D}} \frac{\overline{\phi(z)} - \overline{\phi(w)}}{(1-\overline{w}z)^{\alpha+2}} f(w) dA_\alpha(w), \quad z \in \mathbb{D}. \end{aligned}$$

Axler ('86) proved that $H_{\overline{\phi}}$ is bounded on A^2_α if and only if ϕ belongs to the Bloch space $\mathcal{B}$

$$\mathcal{B} := \{\phi \in \mathrm{Hol}(\mathbb{D}) : \sup_{|z|<1} (1 - |z|^2)|\phi'(z)| < \infty\}.$$

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And $H_{\overline{\phi}}$ is compact on A^2_α if and only if ϕ belongs to the little Bloch space $\mathcal{B}_0$

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It is also easy to see that $H_{\overline{\phi}}$ is a Hilbert Schmidt operator if and only if $\phi \in \mathcal{D}$, where

$$\mathcal{D} := \{\phi \in \mathrm{Hol}(\mathbb{D}) : \phi' \in L^2(\mathbb{D})\},$$

is the Dirichlet space.

Idea of the proof :

The function $u_f(z) = \int_{\mathbb{D}} f(w) \frac{(1-|w|^2)^{1+\alpha}}{(1-\bar{z}w)^{1+\alpha}} dA(w)$ satisfies $\bar{\partial} u_f = f$.

We have

$$\bar{\partial} H_{\bar{\phi}}(f) = \bar{\partial}(\bar{\Phi} f - P_\alpha(\bar{\Phi})) = \overline{\phi'} f \qquad (\star),$$

and $H_{\bar{\phi}}(f)$ is the minimal solution (in L^2_α) of $(\star)$. Then

$$\|H_{\bar{\phi}}(f)\|^2 \lesssim \int_\Omega |f(z)|^2 |\phi'(z)|^2 (1-|z|^2)^2 (z) dA_\alpha(z) = \|J_{\mu_\phi} f\|^2,$$

where

$$d\mu_\phi(z) = (1-|z|^2)^2 |\phi'(z)|^2 dA_\alpha(z),$$

and J_{μ_ϕ} is the embedding operator from A^2_α into $L^2(\mu_\phi)$.

For the converse, it suffices to remark that

$$(H_{\bar{\phi}} K_a)(z) = (\bar{\phi}(z) - \bar{\phi}(a)) K_a(z), \quad z, a \in \Omega.$$

1988, Arazy, Fisher and Peetre proved that if $p > 1$, $H_{\bar{\phi}} \in S_p$ if and only if $\phi \in \mathcal{B}_p$, where

$$\mathcal{B}_p := \{\phi \in \mathrm{Hol}(\mathbb{D}) : \int_{\mathbb{D}} (1-|z|^2)^p |\phi'(z)|^p \frac{dA(z)}{(1-|z|^2)^2} < \infty\}$$

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On the other hand they proved that if $\phi'' \in L^1(dA)$ then

$$\sigma_n(H_{\overline{\phi}}) := \sum_{k=1}^{n} s_k(H_{\overline{\phi}}) = O(\log(n+2)).$$

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In this talk we are interested in the behavior of the singular values of $H_{\bar{\Phi}}$;

$$s_n(H_{\bar{\Phi}}) \asymp ? \text{ (in terms of } \Phi).$$

Weighted Bergman spaces

Let Ω be a domain of $\mathbb{C}$. We denote by $\mathrm{Hol}(\Omega)$ the class of all holomorphic functions on Ω.

Let $\omega : \ \Omega \to (0, \infty)$ be a continuous weight on Ω.

The weighted Bergman space associated with ω is given by

$$\mathrm{A}^2_\omega = \{f \in \mathrm{Hol}(\Omega) : \ \ \|f\|_\omega = \left(\int_\Omega |f(z)|^2 dA_\omega(z) \right)^{1/2} < \infty\},$$

where $dA_\omega(z) = \omega(z) dA(z)$.

A^2_ω is a reproducing Kernel space. The Kernel of A^2_ω will be denoted by K.

The Hankel operator $H_{\overline{\phi}}$, acting on A^2_ω, induced by $\phi \in Hol(\Omega)$ is given by

$$H_{\overline{\phi}}(f) = \overline{\phi} f - P_\omega(\overline{\phi} f).$$

We suppose in the sequel that $H_{\overline{\phi}}$ is densely defined on A^2_ω.

The class of weights $\mathcal{W}$.

Let Ω be a domain (bounded or not) of $\mathbb{C}$ and let $\partial\Omega$ denotes the boundary of Ω.

Let $\partial_\infty\Omega = \partial\Omega$ if Ω is bounded and $\partial_\infty\Omega = \partial\Omega \cup \{\infty\}$ if Ω is not bounded.

In what follows, we suppose that the kernel K of A^2_ω satisfies

$$\lim_{z\to\partial_\infty\Omega} \|K_z\| = \infty \tag{1}$$

$$\text{For every } \zeta \in \Omega, \ \ |K(\zeta,z)| = o(\|K_z\|) \qquad (z \to \partial_\infty\Omega). \tag{2}$$

Let

$$\tau^2(z)(=\tau^2_\omega(z)) := \frac{1}{\omega(z)\|K_z\|^2}, \quad (z \in \Omega).$$

We say that $\omega \in \mathcal{W}$ if, in addition, there exist two constants $a, C > 0$ such that for $z, \zeta \in \Omega$ satisfying $|z-\zeta| \leq a\tau_\omega(z)$ we have :

$$\|K_z\|\|K_\zeta\| \leq C|K(\zeta,z)|, \quad \frac{1}{C}\tau(\zeta) \leq \tau(z) \leq C\tau(\zeta). \tag{3}$$

and

$$\tau(z) = O(\min(1, dist(z, \partial_\infty\Omega))). \tag{4}$$

Examples

- Standard Bergman spaces on the unit disc $\mathbb{D}$.

$$\mathrm{A}^2_\alpha := \left\{ f \in H(\mathbb{D}) : \ \|f\|^2_\alpha = \int_{\mathbb{D}} |f(z)|^2 \, dA_\alpha(z) < \infty \right\}, \text{ where } \ (\alpha > -1).$$

The reproducing kernel is given by

$$K^\alpha_z(w) = \frac{1}{(1 - z\overline{w})^{2+\alpha}}, \qquad \tau(z) \asymp 1 - |z|^2.$$

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- Weighted Bergman spaces on $\mathbb{D}$. Let $\omega = e^{-\varphi}$ be such that φ is sub-harmonic weight on $\mathbb{D}$ and $\frac{1}{\sqrt{\Delta\varphi}}$ satisfies a Lipschitz condition. Then $\omega \in \mathcal{W}$,

$$\tau^2_\omega(z) = \frac{1}{\|K^\omega_z\|^2 \omega(z)} \asymp \frac{1}{\Delta\varphi(z)}.$$

(Hu, Lv and Schuster, J. F. A. (2019) and references therein)

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- Harmonically weighted Bergman spaces on $\mathbb{D}$. Let $\omega > 0$ and harmonic. One can prove that $\omega \in \mathcal{W}$ and

$$\tau(z) \asymp 1 - |z|^2, \ z \in \mathbb{D}$$

(O.E, I. Marhrhich, H. Mahzouli and H. Naqos J.M.A.A 2018)

Theorem

Let $\omega \in \mathcal{W}$. *We have*

1. $H_{\overline{\phi}}$ *is bounded on* A^2_ω *if and only if* $\sup_{z\in\Omega} \tau(z)|\phi'(z)| < \infty$.
2. $H_{\overline{\phi}}$ *is compact on* A^2_ω *if and only if* $\lim_{z\in\Omega\to\partial_\infty\Omega} \tau(z)|\phi'(z)| = 0$.
3. *Let* $p \geq 1$. *Then* $H_{\overline{\phi}} \in S_p(\mathrm{A}^2_\omega)$ *if and only if*

$$\int_\Omega |\phi'(z)|^p \tau^{p-2}(z)\, dA(z) < \infty.$$

Starting point

Lemma

Let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ be two decreasing sequences such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$ and that $(n^\gamma b_n)$ is increasing for some $\gamma \in (0,1)$.

Suppose that there exists $B > 0$ such that

$$\sum_{n\geq 1} h(b_n/B) \leq \sum_{n\geq 1} h(a_n) \leq \sum_{n\geq 1} h(Bb_n),$$

for all increasing convex function h. Then

$$a_n \asymp b_n.$$

That is, there exists $C > 0$ such that

$$b_n/C \leq a_n \leq Cb_n.$$

We will use the lemma as follows.

Lemma

Let $p \geq 1$ *and let* $\rho : [0, +\infty] \to [1, +\infty[$ *be an increasing positive function such that* $\rho(x)/x^{\gamma}$ *is decreasing for some* $\gamma \in (0, p)$. *Let* T *be a positive compact operator. Suppose that there exists* $B > 0$ *such that*

$$\sum_{n\geq 1} h\left(\frac{1}{B\rho(n)}\right) \leq Tr\,(h(T)) = \sum_{n\geq 1} h(\lambda_n(T)) \leq \sum_{n\geq 1} h\left(\frac{B}{\rho(n)}\right),$$

for all increasing functions h such that $h(t^p)$ *is convex. Then*

$$\lambda_n(T) \asymp 1/\rho(n).$$

Toeplitz operators

The Toeplitz operator T_μ, acting on A^2_ω, induced by a positive Borel measure μ on Ω is given by

$$T_\mu f(z) = \int_\Omega f(\zeta)K(z,\zeta)\omega(\zeta)d\mu(\zeta).$$

Note that

$$\langle T_\mu f, f\rangle = \int_\Omega |f(\zeta)|^2\omega(\zeta)d\mu(\zeta).$$

It is known that for $\omega \in \mathcal{W}$, there exist $B, \delta > 0$ and $(z_n)_n \subset \Omega$ such that

- $(D(z_n, \delta\tau_\omega(z_n)))_n$ is a covering of Ω of finite multiplicity.
- $D(z_n, \frac{\delta}{B}\tau_\omega(z_n))$ are pairwise disjoint.

Such family $(D(z_n, \delta\tau_\omega(z_n)))_n$ is called a Lattice of Ω with respect to ω.

Boundedness and compactness :

Fix a lattice $(R_n)_n$ of Ω with respect to ω. One can see that

- T_μ is bounded $\iff \mu(R_n)/A(R_n)$ is bounded.
- T_μ is compact $\iff \mu(R_n)/A(R_n) \to 0$.

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The key of the proof is the following mean inequality

$$|f(\zeta)|^2\omega(\zeta) \lesssim \frac{1}{A(R_n)} \int_{bR_n} |f(z)|^2\omega(z)dA(z) \qquad (\zeta \in R_n, b > 1),$$

with the notation $(bD(z,r) = D(z,br))$.

If T_μ is compact, $(a_n(\mu))_n$ will denote a decreasing rearrangement of $(\mu(R_n)/A(R_n))_n$.

Trace estimates for Toeplitz operators

Theorem (O. E. and M. Elibbaoui 2018)

Let μ be a positive Borel measure on Ω s.t. T_μ is compact on A^2_ω. Let h be increasing, $h(0) = 0$ and $h(t^p)$ is convex for some $p \geq 1$. Then, there exists $B > 1$, which depends on ω and p s.t.

$$\sum_n h\left(\frac{1}{B} a_n(\mu)\right) \leq \sum_n h(\lambda_n(T_\mu)) \leq \sum_n h(B a_n(\mu)).$$

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$$\sum_n h\left(\frac{1}{B} a_n(\mu)\right) \leq \sum_n h(\lambda_n(T_\mu)) \leq \sum_n h(Ba_n(\mu)).$$

As consequence we obtain

Theorem

Let $A > 0$ and let ρ be an increasing positive function s.t. $\rho(x)/x^A$ is decreasing for some $A > 0$. Let μ be a positive Borel measure on Ω such that T_μ is compact. Then

1. $\lambda_n(T_\mu) = O(1/\rho(n)) \iff a_n(\mu) \asymp O(1/\rho(n))$.
2. $\lambda_n(T_\mu) \asymp 1/\rho(n) \iff a_n(\mu) \asymp 1/\rho(n)$.

Remarks

- The growth condition on ρ is, in some sense, necessary. Indeed, let

$$\rho_{\beta,c}(n) = \exp(-c\log^{\beta} n) \quad \beta, c > 0.$$

From Theorem A, it is clear that if $\beta \le 1$ then

$$\lambda_n(T_\mu) \asymp 1/\rho_{\beta,c}(n) \iff a_n(\mu) \asymp 1/\rho_{\beta,c}(n).$$

While, if $\beta > 1$ one can construct a Toeplitz operator T_μ such that

$$\lambda_n(T_\mu) \asymp 1/\rho_{\beta,c}(n) \quad \text{and} \quad \limsup_{n\to\infty} \frac{\lambda_n(T_\mu)}{a_n(\mu)} = +\infty.$$

- One can construct two positive Borel measures μ and ν on the unit disc $\mathbb{D}$ such that

$$a_n(\mu) = a_n(\nu) \quad \text{and} \quad \limsup_{n\to\infty} \lambda_n(T_\mu)/\lambda_n(T_\nu) = \infty.$$

So, it is somewhat surprising that the behavior of $\lambda_n(T_\mu)$, in our case, depends only on $a_n(\mu)$ and not on the positions of $(R_n(\mu))$.

Trace estimates for Hankel operators

Notations :

$$d\lambda_\omega(z) := \frac{dA(z)}{\tau_\omega^2(z)}, \qquad |\,T\,| = (T^*T)^{1/2}.$$

Theorem

Suppose that $H_{\bar{\phi}}$ is compact and let h be an increasing convex function such that $h(0) = 0$. Then

$$\int_\Omega h\left(\frac{1}{B}|\phi'(z)|\tau_\omega(z)\right) d\lambda_\omega(z) \leq Tr\Big(h(|H_{\bar{\phi}}|)\Big) \leq \int_\Omega h\left(B|\phi'(z)|\tau_\omega(z)\right) d\lambda_\omega(z),$$

where the constant $B > 0$ depends only on ω.

Idea of the proof :

Upper estimate : $$\mathrm{Tr}\left(h(|H_{\overline{\phi}}|)\right) \leq \int_{\mathbb{D}} h\left(B|\phi'(z)|\tau_\omega(z)\right) d\lambda_\omega(z).$$

We have

$$\overline{\partial} H_{\overline{\phi}}(f) = \overline{\partial}(\bar{\Phi} f - P_\omega(\bar{\Phi})) = \overline{\phi'} f \qquad (\star).$$

Then $H_{\overline{\phi}}(f)$ is the minimal solution (in L^2_ω) of $(\star)$. Hörmander type estimates for $\overline{\partial}-$equation , imply

$$\|H_{\overline{\phi}}(f)\|^2 \lesssim \int_\Omega |f(z)|^2 |\phi'(z)|^2 \tau^2_\omega(z) dA_\omega(z).$$

This means that

$$H^*_{\overline{\phi}} H_{\overline{\phi}} \lesssim T_{\mu_\phi}, \quad \text{where } d\mu_\phi(z) = \tau^2_\omega |(z)\phi'(z)|^2 dA_\omega(z).$$

Then

$$s^2_n(H_{\overline{\phi}}) \lesssim \lambda_n(T_{\mu_\phi}).$$

Lower estimate : $$\int_{\mathbb{D}} h\left(\frac{1}{B}|\phi'(z)|\tau_\omega(z)\right) d\lambda_\omega(z) \leq \mathrm{Tr}\left(h(|H_{\overline{\phi}}|)\right)$$

We have

$$(H_{\overline{\phi}}K_a)(z) = (\overline{\phi}(z) - \overline{\phi}(a))K_a(z), \quad z, a \in \Omega.$$

Then

$$\begin{aligned} h(\tau_\omega(z)|\phi'(z)|) &\lesssim \int_{bR_n}\int_{bR_n} h(|\phi(\zeta)-\phi(w)|)\, d\lambda_\omega(\zeta) d\lambda_\omega(w). \\ &\lesssim \int_{bR_n}\int_{bR_n} h\left(\frac{|H_{\overline{\phi}}K_\zeta(w)|}{|K(\zeta,w)|}\right) d\lambda_\omega(\zeta) d\lambda_\omega(w) \\ &\lesssim \int_{bR_n}\int_{bR_n} h\left(\frac{|H_{\overline{\phi}}K_\zeta(w)|}{\|K_\zeta\|\|K_w\|}\right) d\lambda_\omega(\zeta) d\lambda_\omega(w). \end{aligned}$$

Let $H_{\overline{\phi}}K_\zeta(w) = \sum_n s_n \overline{f_n(\zeta)} g_n(w)$. We have

$$h\left(\frac{|H_{\overline{\phi}}K_\zeta(w)|}{\|K_\zeta\|\|K_w\|}\right) \leq h\left(\sum_k s_k \frac{|f_k(\zeta)|}{\|K_\zeta\|}\frac{|g_k(w)|}{\|K_w\|}\right) \leq \frac{1}{2}\sum_k \left(\frac{|f_k(\zeta)|^2}{\|K_\zeta\|^2} + \frac{|g_k(w)|^2}{\|K_w\|^2}\right) h(s_k).$$

Combining these inequalities, and after integration we obtain the result.

Consequence

Let $\mathrm{R}^+_{\phi,\omega}$ be the decreasing rearrangement of $\tau_\omega|\phi'|$ with respect to λ_ω. Namely,

$$\mathrm{R}^+_{\phi,\omega}(x) := \sup\{t \in (0, \|\tau\phi'\|_\infty] : \ \mathrm{R}_{\phi,\omega}(t) \geq x\},$$

where

$$\mathrm{R}_{\phi,\omega}(t) := \lambda_\omega(\{z \in \Omega : \ \tau_\omega(z)|\phi'(z)| > t\}).$$

Remark that

$$\int_\Omega h\left(|\phi'(z)|\tau_\omega(z)\right) d\lambda_\omega(z) \asymp \sum_n h\left(\mathrm{R}^+_{\phi,\omega}(n)\right)$$

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Theorem

Let ρ *be an increasing function such that* $\rho(x)/x^\gamma$ *is decreasing for some* $\gamma \in (0,1)$ *then*

$$s_n(H_{\overline{\phi}}) \asymp 1/\rho(n) \quad \Longleftrightarrow \quad \mathrm{R}^{+}_{\phi,\omega}(n) \asymp 1/\rho(n).$$

Radial weighted Bergman spaces on the unit disc

The singular values of $H_{\bar{z}}$ will play an important role in the study of the decay of singular values of Hankel operators with anti-analytic symbols. Note that in the radial case we have

$$H_{\bar{z}}^{*}H_{\bar{z}}\left(\frac{z^n}{\|z^n\|}\right)=\left(\frac{\|z^{n+1}\|^2}{\|z^n\|^2}-\frac{\|z^n\|^2}{\|z^{n-1}\|^2}\right)\frac{z^n}{\|z^n\|}=: m_\omega(n)\frac{z^n}{\|z^n\|},\quad n\geq 1.$$

So, the sequence of the singular values of $H_{\bar{z}}$ is exactly the sequence $(\sqrt{m_\omega(n)})_{n\geq 1}$.

The standard Bergman spaces A^2_α, which correspond to $\omega_\alpha(z)=(1+\alpha)(1-|z|^2)^\alpha$, with $\alpha>-1$. In this case, we have

$$\|z^n\|^2_{\omega_\alpha}=\frac{\Gamma(n+1)\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)},$$

and

$$m_{\omega_\alpha}(n)=\frac{\alpha+1}{(n+\alpha)(n+\alpha+1)}\sim\frac{\alpha+1}{(n+1)^2}.$$

Then

$$s_n(H_{\bar{z}})\sim\frac{\sqrt{\alpha+1}}{n+1}.$$

Let $\beta \geq 0$ and let ω be such that $\tau_\omega^2(z) \asymp (1-|z|^2)^{2+\beta}$, one can consider for example

$$\omega(z) = (1-|z|^2)^\alpha \exp\left(-(1-|z|^2)^{-\beta}\right), \quad \alpha > -1.$$

We have

$$\begin{aligned} \mathrm{R}_{z,\omega}(t) &= \lambda_\omega\{z \in \mathbb{D} : \ \tau^2(z) \geq t\} \\ &\asymp \int_{\{z\in\mathbb{D}: \ \tau^2(z)\geq t\}} \frac{dA}{(1-|z|^2)^{2+\beta}} \\ &\asymp \int_{\{r\in(0,1): \ (1-r)^{2+\beta}\geq t\}} \frac{dr}{(1-r)^{2+\beta}} \\ &\asymp t^{-\frac{2(1+\beta)}{2+\beta}}. \end{aligned}$$

Then

$$\mathrm{R}^+_{z,\omega}(t) \asymp \frac{1}{t^{1/p}}, \quad \text{where } p = \frac{2(1+\beta)}{2+\beta},$$

and

$$\sqrt{m_\omega(n)} = s_n(H_{\bar{z}}) \asymp \frac{1}{n^{1/p}}.$$

We have the following result.

Theorem

Let $\omega \in \mathcal{W}$ be a radial weight. Let ϕ be an analytic function such that $H_{\overline{\phi}}$ is compact. Then

$$s_n(H_{\overline{\phi}}) = o(s_n(H_{\overline{z}})) \quad \Longrightarrow \quad H_{\overline{\phi}} = 0.$$

Suppose that $\tau_\omega(z) \asymp (2 - |z|^2)^{2+\beta}$, then

$$s_n(H_{\overline{\phi}}) = o(1/n^{1/p}) \quad \Longrightarrow \quad H_{\overline{\phi}} = 0.$$

Here $p = \frac{2(1+\beta)}{2+\beta}$.

Now we are interested in the description of the class of symbols ϕ such that

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Theorem

Let $\beta \geq 0$ *and let* ω *such that* $\tau_\omega^2(z) \asymp (1-|z|^2)^{2+\beta}$. *Then,*

- $$s_n(H_{\overline{z}}) \asymp 1/n^{1/p}, \quad p = \frac{2(1+\beta)}{2+\beta}.$$
- $$s_n(H_{\overline{\phi}}) = O(1/n^{1/p}) \quad \Longleftrightarrow \quad \phi' \in H^p.$$

Idea of the proof of : $\phi' \in H^p \implies s_n(H_{\bar{\phi}}) = O(1/n^{1/p})$.

It suffices to prove that $\mathrm{R}^+_{\Phi,\omega}(x) = O(1/x^p)$. That is $\mathrm{R}_{\Phi,\omega}(t) = O(1/t^p)$.

Let U be the non tangential maximal function associated with $|\phi'|$. It satsfies

$$|\phi'(re^{i\theta})| \leq U(e^{i\theta}) \; a.e. \text{ on } \mathbb{T} \text{ and } U \in L^p.$$

Then

$$\begin{aligned}
\mathrm{R}_{\Phi,\omega}(t) &= \lambda_\omega(\{re^{i\theta} \in \mathbb{D} : \tau_\omega(r)|\phi'(re^{i\theta})| \geq t\}) \\
&\leq \lambda_\omega(\{re^{i\theta} \in \mathbb{D} : \tau_\omega(r)|U(e^{i\theta})| \geq t\}) \\
&= \int_{\{re^{i\theta} : \tau_\omega(r)|U(e^{i\theta})| \geq t\}} \frac{dr}{\tau^2_\omega(r)} d\theta \\
&\leq \int_{\mathbb{T}} \int_{\{r : (1-r)^{1+\beta/2} \geq C \frac{t}{|U(e^{i\theta})|}\}} \frac{dr}{(1-r)^{2+\beta}} d\theta \asymp \frac{\|U\|_p^p}{t^p}.
\end{aligned}$$

Assymptotics

Pb : Is it possible to obtain the exact asymptotic behavior of $s_n(H_{\overline{\phi}})$ for some particular ϕ ?

First let $\phi(z) = z$. For a radial weight ω, the sequence of the singular values of $H_{\overline{z}}$ is

$$\left(\left(\frac{||z^{n+1}||^2}{||z^n||^2} - \frac{||z^n||^2}{||z^{n-1}||^2}\right)^{1/2}\right)_n .$$

- The standard Bergman space $\omega_\alpha(z) = (1-|z|^2)^\alpha$:

$$s_n(H_{\overline{z}}) \sim \frac{\sqrt{\alpha+1}}{n+1},\ n \to \infty.$$

- For the weight ω given by

$$\omega(z) = \exp\left(-\frac{\alpha}{(\log \frac{1}{|z|^2})^\beta}\right), \quad \alpha,\ \beta > 0.$$

Recall that $\tau_\omega^2(z) \asymp (1-|z|^2)^{2+\beta}$ and $s_n(H_{\overline{z}}) \asymp \frac{1}{n^{1/p}}$, where $p = \frac{2(1+\beta)}{2+\beta}$.

$$||z^n||^2 = \int_{\mathbb{D}} |z|^{2n} \exp\left(-\frac{\alpha}{(\log\frac{1}{|z|^2})^\beta}\right) dA(z) = \int_0^1 r^{2n} \exp\left(-\frac{\alpha}{(\log\frac{1}{r^2})^\beta}\right) 2rdr$$
$$= \int_0^{+\infty} \exp\left(-(n+1)x - \frac{\alpha}{x^\beta}\right) dx.$$

Let $x_n := \left(\frac{\alpha\beta}{n+1}\right)^{1/1+\beta}$ be the minimum of the function $(n+1)x + \frac{\alpha}{x^\beta}$. After the change of variable $u = \frac{x-x_n}{x_n}$, we get

$$||z^n||^2 = x_n \exp\left(-(n+1)x_n - \frac{\alpha}{x_n^\beta}\right) \int_{-1}^{+\infty} \exp\left(-\frac{\alpha}{x_n^\beta} h(u)\right) du,$$

where $h(u) = \beta u + \frac{1}{(1+u)^\beta} - 1$. Then Laplace Theorem, gives

$$\int_{-1}^{+\infty} \exp\left(-th(u)\right) du \sim \sqrt{\frac{2\pi}{th''(0)}}, \quad t \to +\infty.$$

Finally, we obtain

$$s_n(H_{\bar{z}}) \sim \frac{\gamma}{n^{\frac{\beta+2}{2(\beta+1)}}}, \quad \text{where } \gamma = \sqrt{\frac{(\alpha\beta)^{1/1+\beta}}{1+\beta}}.$$

Theorem

Let $\beta \geq 0$ *and let* $\omega \in \mathcal{W}$ *be a radial weight such that* $\tau_\omega^2(z) \asymp (1-|z|^2)^{2+\beta}$. *Then,*

$$s_n(H_{\overline{\phi}}) = O(1/n^{1/p}) \quad \Longleftrightarrow \quad \phi' \in H^p \qquad \left(p = \frac{2(1+\beta)}{2+\beta}\right)$$

Moreover, if $s_n(H_{\bar{z}}) \sim \frac{c}{n^{\frac{1}{p}}}$ *then*

$$s_n(H_{\overline{\phi}}) \sim \frac{c}{n^{1/p}} \|\phi'\|_p.$$

Theorem

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Moreover, if $s_n(H_{\bar z}) \sim \frac{c}{n^{\frac{1}{p}}}$ *then*

$$s_n(H_{\bar\phi}) \sim \frac{c}{n^{1/p}} \|\phi'\|_p.$$

In the case of the classical Bergman space A^2 ($\alpha = 0$).

- M. Dostanic (2004) proved that if ϕ is analytic in a neighborhood of $\bar{\mathbb{D}}$, then $s_n(H_{\bar\phi}) \sim \frac{\|\phi'\|_1}{n}$.
- Englis and Rochberg (2009) proved that if $\phi' \in H^1$, then $H_{\bar\phi}$ is in the Dixmier class and that the Dixmier trace is given by
$$\mathrm{Tr}(|\, H_{\bar\phi} \,|) = \|\phi'\|_1.$$

The proof uses some ideas of M. Dostanic's proof, a result on asymptotic spectral orthogonality due to Birman and Solomyak (see the paper by A. Pushnitski : Spectral asymptotics for Toeplitz operators and an application to banded matrices, (2018)) and the theorem of trace estimates of Hankel operators.

Thank you for your attention.